Exercise 18

Write a trial solution for the method of undetermined coefficients. Do not determine the coefficients.

$$y'' + 4y = e^{3x} + x\sin 2x$$

Solution

Since the ODE is linear, the general solution can be written as the sum of a complementary solution and a particular solution.

$$y = y_c + y_p$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 4y_c = 0 \tag{1}$$

This is a linear homogeneous ODE, so its solutions are of the form $y_c = e^{rx}$.

$$y_c = e^{rx} \quad \rightarrow \quad y_c' = re^{rx} \quad \rightarrow \quad y_c'' = r^2e^{rx}$$

Plug these formulas into equation (1).

$$r^2e^{rx} + 4(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 4 = 0$$

Solve for r.

$$r = \{-2i, 2i\}$$

Two solutions to the ODE are e^{-2ix} and e^{2ix}. By the principle of superposition, then,

$$\begin{aligned} y_c(x) &= C_1e^{-2ix} + C_2e^{2ix} \\ &= C_1(\cos 2x - i\sin 2x) + C_2(\cos 2x + i\sin 2x) \\ &= (C_1 + C_2)\cos 2x + (-iC_1 + iC_2)\sin 2x \\ &= C_3\cos 2x + C_4\sin 2x. \end{aligned}$$

On the other hand, the particular solution satisfies the original ODE.

$$y_p'' + 4y_p = e^{3x} + x\sin 2x$$

Since the inhomogeneous term is the sum of an exponential and a polynomial of degree 1 times sine, the particular solution would be

$$y_p = Ae^{3x} + (Bx + C)(D\sin 2x + E\cos 2x).$$

$\sin 2x$ and $\cos 2x$ already satisfy the complementary solution, though, so an extra factor of x is needed.

$$y_p = Ae^{3x} + x(Bx + C)(D\sin 2x + E\cos 2x)$$